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STABILITY AND BIFURCATION IN A PARABOLIC EQUATION.(U)

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REPORT DOCUMENTATION PAGE	READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 80-1148 — 2. GOVT ACCESSION NO. AD-A091645	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) STABILITY AND BIFURCATION IN A PARABOLIC EQUATION	5. TYPE OF REPORT & PERIOD COVERED Interim
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) JACK K. HALE	8. CONTRACT OR GRANT NUMBER(s) AFOSR 76-3092
9. PERFORMING ORGANIZATION NAME AND ADDRESS DIVISION OF APPLIED MATHEMATICS BROWN UNIVERSITY PROVIDENCE, RHODE ISLAND 02912	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2304/A1
11. CONTROLLING OFFICE NAME AND ADDRESS AIR FORCE OFFICE OF SCIENTIFIC RESEARCH BOLLING AIR FORCE BASE WASHINGTON, D.C.	12. REPORT DATE AUGUST 22, 1980
	13. NUMBER OF PAGES 19
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

DTIC ELECTE NOV 13 1980 S D E

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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(14) AFOSR-TR-80-1148 (19)

(6) STABILITY AND BIFURCATION IN A PARABOLIC EQUATION.*

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(16) 2311

(17) A1

(11) 22 Aug[redacted]80

(12) 20

(15) DAAG27-79-C-0161, AFOSR-76-3

*This research was supported in part by the National Science Foundation under MCS-79-0-774, in part by the United States Army under AROD DAAG 27-79-C-0161, and in part by the United States Air Force under [redacted]AFOSR 76-3092[redacted].

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STABILITY AND BIFURCATION IN A PARABOLIC EQUATION

by

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Abstract

Recent results on the stability of equilibrium solutions of a parabolic equation are given with indications of the proofs. Particular attention is devoted to dependence of the stability properties on the shape of the domain and the manner in which nonhomogeneous stable equilibria can occur through a bifurcation induced by varying the domain.

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In this paper, we present a few recent results on the asymptotic behavior of the solutions of a parabolic equation of the form

$$
\begin{aligned}
&u_t = \Delta u + f(u) \quad \text{in} \quad \Omega \\
&\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega
\end{aligned}
\tag{1}
$$

where Ω is a bounded open set in $\mathbb{R}^n$ with smooth boundary. We also discuss how the qualitative behavior of the stable equilibria depend upon the shape of Ω and the nonlinear function f. The function f is supposed to satisfy conditions which ensure that Eq. (1) defines a strongly continuous semigroup $T_f(t)$ on $H^1(\Omega)$.

We remark that the boundary conditions in (1) are not important as far as the spirit of the problems to be discussed. Of course, the details will depend generally in a very significant manner upon the boundary conditions.

Let $E_f(\Omega)$ be the set of equilibrium solutions of (1); that is, the set of solutions of the equation

$$
\begin{aligned}
&\Delta u + f(u) = 0 \quad \text{in} \quad \Omega \\
&\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \, .
\end{aligned}
\tag{2}
$$

If $\Omega = (0,\lambda) \subseteq \mathbb{R}$, then the set $E_f(0,\lambda)$ is given by

$$E_f(0,\lambda) = \{\text{periodic solutions of period } 2\lambda \text{ of the equation } u_{xx} + f(u) = 0\} \tag{3}$$

In the following, we let $W^u(\phi)$, $W^s(\phi)$ denote, respectively, the unstable and stable manifolds for an equilibrium point ϕ of (1). The following result for $n = 1$ is due to a number of authors. The references are in the proof.

Theorem 1. (n=1) If $\Omega = (0,\lambda) \subset \mathbb{R}$, then

(i) the ω-limit set of any bounded solution of (1) is in $E_f(0,\lambda)$.

(ii) the ω-limit set of any bounded solution of (1) is a single point in $E_f(0,\lambda)$.

(iii) the only stable equilibrium points of (1) are constants.

(iv) If $\int_0^u f(s)ds \to -\infty$ as $u \to \pm\infty$, then every solution of (1) is bounded. If, in addition, $E_f(0,\lambda)$ is a bounded set, then there is a maximal compact invariant set $A_f(0,\lambda)$ of (1),

$$A_f(0,\lambda) = \cup_{\phi \in E_f(0,\lambda)} W^u(\phi),$$

$A_f(0,\lambda)$ is uniformly asymptotically stable and, for any bounded set B in $H^1(\Omega)$, $\operatorname{dist}(T_f(t)B, A_f(0,\lambda)) \to 0$ as $t \to \infty$.

(v) If, in addition to the hypothesis in (iv), all $\phi \in E_f(0,\lambda)$ are hyperbolic, then, for any bounded set B in $H^1(\Omega)$, the set

$$(\bigcup_{\phi \text{ stable}} W^s(\phi)) \cap B$$

is open and dense in B.

Proof: (i) This is due to Chafee [4] and is independent of the boundary conditions. The idea is very simple.

If

$$V(u) = \int_0^\lambda (u_x^2 - \int_0^u f(s)ds)dx$$

then the derivative of V along the solutions of (1) satisfies

$$\dot{V}(u) = -\int_0^\lambda u_t^2 \, dx \leq 0.$$

Since every bounded orbit is precompact, a simple application of the invariance principle implies the result.

(ii) This result is due to Matano [13] and is independent of the boundary conditions. He used a rather sophisticated application of the invariance principle. We sketch a proof based on the theory of dynamical systems. The details will appear in Hale and Massatt [6]. The idea is very simple and can be traced to Malkin [12], Hale and Stokes [7] and perhaps even further.

If ϕ is an element of an ω-limit of an orbit which is not a single point, then ϕ belongs to a continuum in $E_f(0,\lambda)$ and ϕ is not hyperbolic. The linear variational equation about ϕ has the simple eigenvalue zero with all other eigenvalues being nonzero. Thus, ϕ belongs to a smooth one dimensional submanifold M of E_f. In modern terminology, ϕ is normally hyperbolic. One now can show that any solution of (1) which remains in a sufficiently small neighborhood of M for t sufficiently large must be of $W^s(M)$, the stable manifold of M. Finally, one shows that each orbit in $W^s(M)$ approaches a single point.

(iii) This result is due to Chafee [4]. The following proof is taken from a preprint of Bardos, Matano and Smoller [1]. Suppose ϕ is a nonconstant equilibrium solution of (1). Then $v = d\phi/dx \not\equiv 0$, $v = 0$ at $x = 0$, $x = \lambda$, and v satisfies the equation

$$v_{xx} + f'(\phi)v = 0.$$

Let σ_N, σ_D be the spectrum of this differential operator with, respectively, homogeneous Neumann and Dirichlet boundary conditions. Since $\inf \sigma_N < \inf \sigma_D$ and $0 \in \sigma_D$, the result is proved.

(iv) Using the function $V(u)$ in part (i), one easily shows that every solution is bounded. Thus, the ω-limit set of every solution of (1) belongs to $E_f(0,\lambda)$. Since $E_f(0,\lambda)$ is

bounded, there is a bounded set B such that every solution of (1) eventually enters B; that is, Eq. (1) is point dissipative. Since the semigroup $T_f(t)$ is compact for $t > 0$, the results follow from the general theory of dissipative processes (see, for example, Hale [9,10]).

(v) This result is due to Henry [11]. The idea of the proof is to observe first that the solution operator for the linear variational equation of (1) about any point is one-to-one. This can then be used to show that, for any $\phi \in E_f(0,\lambda)$ which is not stable, the set $W^s(\phi) \cap B$ is nowhere dense in B for any bounded set B. As remarked by Mañé, the one-to-oneness of this solution operator implies that the stable manifold $W^s(\phi)$ can actually be given globally a manifold structure. This also gives a proof of the assertion in (v).

For $\Omega = (0,\lambda)$, we have remarked that the set $E_f(0,\lambda)$ coincides with the set of 2λ-periodic solutions of $u_{xx} + f(u) = 0$. For any $a \in \mathbb{R}$, let $u(x,a)$ be the solution of this equation with $u(0,a) = a, u_x(0,a) = 0$. If a is such that $u(x,a)$ is periodic in x, let $2\lambda_f(a)$ be the period. The function $u(\cdot,a) \in E_f(0,\lambda_f(a))$. For $f(u)$ an arbitrary cubic polynomial in u, Smoller and Wasserman [15] have shown that the function $\lambda_f(a)$ has a finite number (≤ 2) of maxima and minima and the second derivative of λ at these points is different from zero; that is, $\lambda_f(a)$ is a Morse function.

The above result has important implications for the applications. In fact, for f any cubic polynomial in u, and for λ fixed and different from the maxima and minima of the function $\lambda_f(a)$, the set $E_f(0,\lambda)$ consists only of hyperbolic points. For λ equal to one of the extreme values of $\lambda_f(a)$, there is a bifurcation of the saddle-node type.

The following qualitative result of Brunovsky and Chow [2] has recently been proved.

Theorem 2. There is a residual set $\mathscr{F} \in \mathscr{C}^2(\mathbb{R})$ with the Whitney topology such that, for any $f \in \mathscr{F}$, the function $\lambda_f(a)$ above is a Morse function.

The proof is not trivial because the function f depends only on u and not on (x,u). The proof is based on a detailed analysis of an analytic expression of $\lambda_f(a)$ as a function of f,a. It is not a trivial exercise because there is no simple way to determine the qualitative properties of the derivatives of this function in a from the derivatives of f. In fact, there are nonlinear functions f for which $\lambda_f(a)$ is constant (see, for example, Urabe [16]).

Theorem 2 can be appropriately generalized to other boundary conditions (see Brunovsky and Chow [2]). Smoller and Wasserman [15] have also considered other boundary conditions.

Our next objective is to discuss the extent to which the above results are valid for a bounded set Ω in $\mathbb{R}^n$.

Theorem 3. If Ω is a bounded set in $\mathbb{R}^n$ with smooth boundary, then

(i) the ω-limit set of a bounded orbit is in $E_f(\Omega)$.

(ii) the ω-limit set of a bounded orbit is a single point if the following condition is satisfied:

If $\phi \in E_f(\Omega)$ is not hyperbolic and k is the dimension of the null space of the operator $\Delta + f'(\phi)$ in Ω with homogeneous Neumann conditions, then ϕ belongs to a smooth submanifold of dimension k.

(iii) Ω convex implies the only stable equilibrium points are constants.

(iv) Same statement as (iv) in Theorem 1 holds.

(v) Same statement as (v) in Theorem 1 holds.

Proof: The proof of (i), (ii), (iv) and (v) are essentially the same as the corresponding assertions in Theorem 1.

(iii) This result was independently discovered by Casten and Holland [3], Matano [14]. The proof exploits special properties of the Laplacian on convex regions to prove that the linear variational equation has a negative eigenvalue for any nonconstant equilibrium.

An analogue of Theorem 2, as far as hyperbolicity of equilibrium and saddle node bifurcations, is not known for Ω in $\mathbb{R}^n$ and seems to be rather difficult.

We remark that part (iii) of Theorem 3 is also valid for Ω convex and the equations

$$\begin{aligned} u_t &= \Delta u + f(u), \quad \text{in } \Omega \\ v_t &= -g(u,v) \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

provided that the spectrum of the operator $\partial g(\xi,\eta)/\partial v$ belongs to the set $\{z : \mathrm{Re}\, z > 0\}$ uniformly in ξ,η (see Bardos, Matano and Smoller [1]).

The remainder of the discussion centers around the case *when Ω is not convex and the objective is to understand more* about the set of stable equilibrium. Before doing this, we make the important remark that, when Ω is convex, the qualitative structure of the stable equilibria is independent of the non-linearity f. When Ω is not convex, this will no longer be the case.

The following result is due to Matano [14].

<u>Theorem 4.</u> There is a nonlinear function f and $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, such that (1) has a stable nonconstant equilibrium.

<u>Idea of proof:</u> Suppose f has zeros only at $a < 0 < b$, they are simple and $\int_0^u f(s)ds \to -\infty$ as $u \to \pm\infty$. Suppose Ω has the

shape shown in Figure 1 and let λ_2 be the minima of the

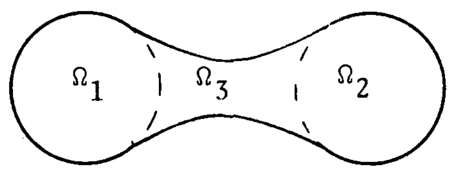

Figure 1

second eigenvalues of the Laplacian on Ω_1 and Ω_2. Matano [ ] gives a specific continuous function $G: \mathbb{R}^2 \to \mathbb{R}$ such that the set where $G < 0$ is nonempty with the property that, if λ_2, Ω_3 are such that $G(\lambda_2, \text{meas } \Omega_3) < 0$, then there is a nonconstant stable equilibrium of (1). For fixed Ω_1, Ω_2, he shows there is an Ω_3 such that the above inequality is satisfied.

The idea of the proof is the following. Let Y be the subset of functions u in $H^1(\Omega)$ such that $\int_{\Omega_1} u > 0$, $\int_{\Omega_2} u < 0$. If meas Ω_3 is small enough, it is then shown that the set Y has a certain invariance property with respect to $T_f(t)$. A careful application of the maximum principle gives a minimal equilibrium solution v_m in Y stable from below and a maximal equilibrium solution v_M in Y stable from above. If it were known that

there are only a finite number of equilibrium solutions, then the proof would be complete. Since this is not known, another argument must be used. Matano first proves that any solution unstable from above must be strongly unstable from above in the sense that it can be isolated from equilibrium solutions from above uniformly in Y. He then puts an ordering on the solutions from above, uses Zorn's lemma and the above property of solutions unstable from above.

In the proof of Theorem 4, the nonlinear function f has the three simple zeros at $a < 0 < b$. The equilibrium points a,b are stable and zero is a saddle point. A stable, nonconstant equilibrium solution was shown to exist. The argument of Matano can be used to show there must be another nonconstant equilibrium solution with $\int_{\Omega_1} u < 0$, $\int_{\Omega_2} u > 0$. Thus, there are at least four stable equilibrium solutions and one unstable equilibrium solution. This is impossible dynamically and there must be some other equilibrium solutions which are unstable. In fact, an index argument implies there must be at least three unstable equilibria. Using more of the detailed information from the paper of Matano, one can show there must be at least five unstable equilibrium solutions. Consequently, there are at least nine equilibrium solutions for this nonlinear function f and region Ω.

The basic problem is to understand in more detail how variations in the shape of the domain Ω cause these additional solutions to appear. We now summarize some work of Hale and Vegas [8] which give a possible explanation.

Let us begin with an intuitive discussion of how the stable nonconstant equilibrium solutions could appear as secondary bifurcations. Suppose $\mu \in [0,\infty)$, Ω_μ is a bounded set in $\mathbb{R}^2$ with smooth boundary with the property that Ω_0 is convex and the second eigenvalue $\lambda_2(\mu)$ of $-\Delta$ on Ω_μ is a monotone decreasing function of μ, approaching zero as $\mu \to \infty$. Also, suppose the third eigenvalue $\lambda_3(\mu)$ of $-\Delta$ on Ω_μ satisfies $\lambda_3(\mu) \geq \delta > 0$ for all μ. Let $f(\nu,u) = \nu^2 u - u^3$, $\nu > 0$, and let $A_{\nu,\mu}$ be the maximal compact invariant set for the equation

$$u_t = \Delta u + f(\nu,u) \quad \text{in } \Omega_\mu$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{in } \partial\Omega_\mu .$$

Fix ν sufficiently small so that $\nu^2 < \delta$ and the only equilibrium solutions in Ω_0 are the constant functions 0, $\pm\nu$. The set $A_{\nu,0}$ is then the constant functions $\pm\nu$ and the unstable manifold of 0, which is one dimensional. Let μ_0 be such that $\lambda_2(\mu_0) = f'(0) = \nu^2$. At the point $\mu = \mu_0$, the zero solution bifurcates creating two new equilibrium solutions which are unstable. They are unstable because the unstable manifold of zero becomes two dimensional - the direction of bifurcation is independent of the direction of the original unstable manifold in $A_{\nu,\mu}$ for $\mu < \mu_0$. The set $A_{\nu,\mu}$ for $\mu > \mu_0$ but close to μ_0 is then two dimensional with three unstable and two stable equilibriums.

Now suppose that Ω_μ has the shape shown in Figure 1 and that meas $\Omega_\mu \to 0$ as $\mu \to \infty$. Then we can find a μ_1 such that the inequality in the proof of Theorem 4 is satisfied. Thus, there is a stable nonconstant equilibrium solution. It is conjectured that this occurs as a secondary bifurcation from the unstable nonconstant equilibria discussed above. In Figures 2 and 3 we have depicted, respectively, the set $A_{\nu,\mu}$ as a function of μ and the conjectured bifurcation diagram.

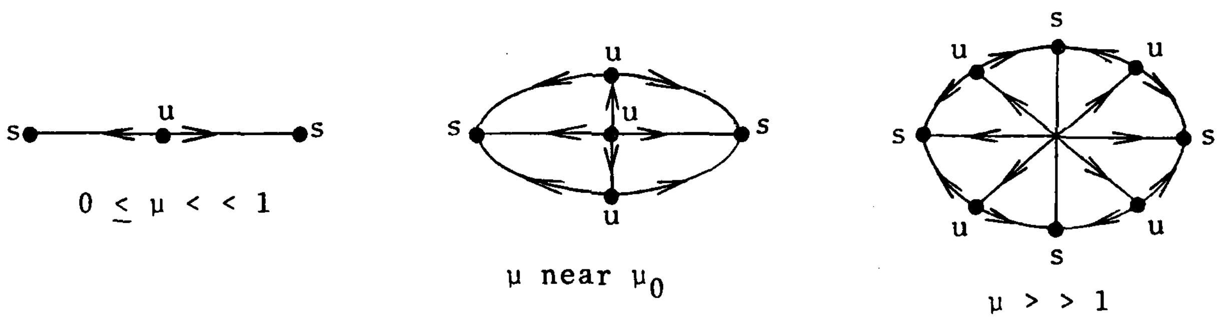

Figure 2

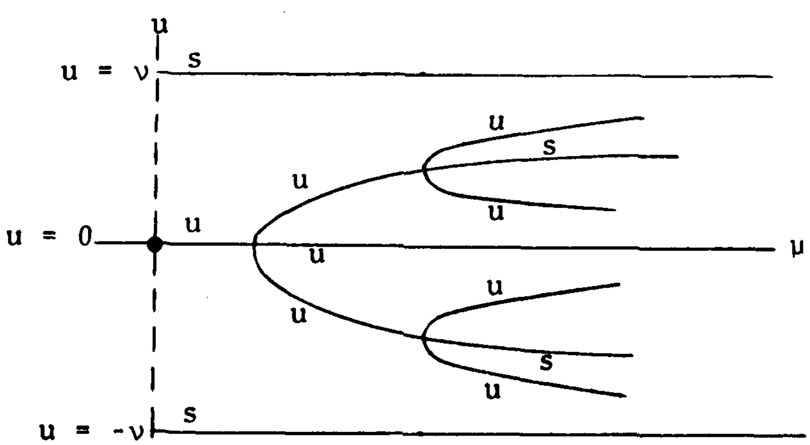

Figure 3

To do things more analytically, we have tried to discuss the neighborhood of $\mu = \infty$ treating it as a bifurcation problem from a double eigenvalue zero. More specifically, suppose $\mu = \varepsilon^{-1}$ and the region Ω_ε is shown in Figure 4, two circles Ω_1, Ω_2 and a small channel between them.

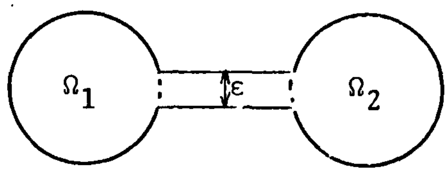

Figure 4

For $\varepsilon = 0$, it is clear there are nine solutions consisting of all combinations of $0, \pm\nu$ on Ω_1 and Ω_2. Five of these are saddles and four are stable nodes. Let $\lambda_2(\varepsilon)$ be the second eigenvalue of $-\Delta$ on Ω_ε and let w_ε be a unit eigenvector corresponding to λ_ε. Let u_ε be the constant function $(\text{meas } \Omega_\varepsilon)^{-1}$; that is, u_ε is a unit eigenvector for the eigenvalue 0 of the Laplacian on Ω_ε.

One can now show that it is possible to apply the method of Liapunov-Schmidt for the solutions of (1) near $u = 0$ for ε,ν near zero. More specifically, for real α,β sufficiently small and ε,ν sufficiently small, there is a function $u^*(\alpha,\beta,\varepsilon,\nu)$ continuously differentiable in α,β,ν and continuous in ε such that $u^*(0,0,0,0) = 0$, $\partial u^*(0,0,0,0)/\partial(\alpha,\beta) = 0$ and

$$\Delta u^* + f(\nu,\alpha u_\varepsilon+\beta w_\varepsilon+u^*) - \pi f(\nu,\alpha u_\varepsilon+\beta w_\varepsilon+u^*) = 0$$

where πu is the projection of u onto the span of the constant function u_ε and the function w_ε; that is,

$$\pi u = \int_{\Omega_\varepsilon} u + w_\varepsilon \int_{\Omega_\varepsilon} w_\varepsilon u.$$

If $u^*(\alpha,\beta,\varepsilon,\nu)$ satisfies the above, then

$$u = \alpha u_\varepsilon + \beta w_\varepsilon + u^*(\alpha,\beta,\varepsilon,\nu)$$

is a solution of (1) if and only if $(\alpha,\beta,\varepsilon,\nu)$ satisfy the bifurcation equations

$$\int_{\Omega_\varepsilon} f(\nu,\alpha u_\varepsilon+\beta w_\varepsilon+u^*(\alpha,\beta,\varepsilon,\nu))dx = 0$$

$$\int_{\Omega_\varepsilon} w_\varepsilon f(\nu,\alpha u_\varepsilon+\beta w_\varepsilon+u^*(\alpha,\beta,\varepsilon,\nu))dx = 0$$

are satisfied.

If we let $(\alpha,\beta) = \gamma$, then one can show that these equations have the form

$$c(\gamma) + \varepsilon L_1\gamma + \nu L_2\gamma + \text{h.o.t.} = 0 \tag{4}$$

where h.o.t. denotes higher order terms in γ,ε,ν, L_1,L_2 are

constant two by two matrices and $c(\gamma)$ is a homogeneous cubic two vector in the components of γ. One can now apply the method in, for example, Chow, Hale and Mallet-Paret [5] to obtain the complete bifurcation diagram for the solutions of (4). These are shown in Fiture 5. Figure 5a shows the number of solutions

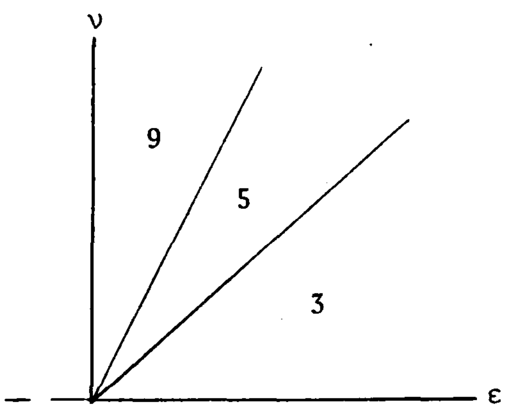

Figure 5a

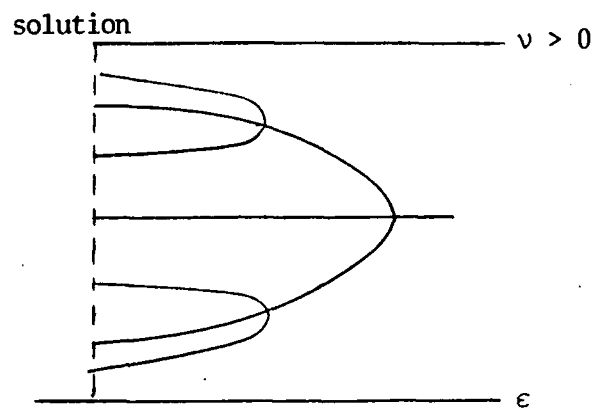

Figure 5b

for a fixed (ε,ν) and Figure 5b shows the way the solutions bifurcation as a function of ε for a fixed ν.

In the verification of the previous results, it is crucial to show that the third eigenvalue of the Laplacian on Ω_ε is bounded away from zero. It would be very interesting to obtain general geometric conditions on a region Ω_ε to have this latter property satisfied.

REFERENCES

[1] C. Bardos, H. Matano and J. Smoller, Some results on the instability of the solutions of reaction diffusion equations. Preprint.

[2] P. Brunovsky and S.N. Chow, Generic properties of stationary states of reaction diffusion equations. To be submitted.

[3] R.G. Casten and C.J. Holland, Instability results for a reaction diffusion equation with Neumann boundary conditions. J. Differential Equations 27(1978), 266-273.

[4] N. Chafee, Asymptotic behavior of a one-dimensional heat equation with homogeneous Neumann boundary conditions. J. Differential Equations 18(1975), 111-134.

[5] S.N. Chow, J.K. Hale and J. Mallet-Paret, Applications of generic bifurcation, II. Arch. Rat. Mech. Anal. 62(1976), 209-236.

[6] J.K. Hale and P. Massatt, The ω-limit set in gradient evolutionary systems. To be submitted.

[7] J.K. Hale and A. Stokes, Behavior of solutions near integral manifolds. Arch. Rat. Mech. Ana. 6(1960), 133-170.

[8] J.K. Hale and J. Vegas, Bifurcation with respect to domain in a parabolic equation. To be submitted.

[9] J.K. Hale, Functional Differential Equations. Appl. Math. Sci. Vol. 3, Springer-Verlag, 1977.

[10] J.K. Hale, Some recent results on dissipative processes. Proc. Symp. on Functional Differential Equations and Dynamical Systems. São Carlos, Brazil, (1979), To appear in Lecture Notes in Math., Springer-Verlag.

[11] D. Henry, Geometric Theory of Semilinear Parabolic Equations. To appear in Lecture Notes in Math., Springer-Verlag.

[12] I. G. Malkin, Theory of Stability of Motion, Moscow, 1952.

[13] H. Matano, Convergence of solutions of one-dimensional semilinear parabolic equations. J. Math. Kyoto University 18(1978), 221-227.

[14] H. Matano, Asymptotic behavior and stability of solutions of semilinear diffusion equations. Res. Inst. Math. Sci., Kyoto 15(1979), 401-454.

[15] J. Smoller and A. Wasserman, Global bifurcation of steady-state solutions. J. Differential Equations. To appear, 12/1980.

[16] M. Urabe, Nonlinear Autonomous Oscillations, Academic Press, 1967.